

8.2

偏导数全微分

同步练习

8.2.一

1. 设 $z = e^{x^2y}$, 求 $\frac{\partial z}{\partial x} =$ _____

解 $\frac{\partial z}{\partial x} = 2xye^{x^2y}$ $\frac{\partial z}{\partial y} = x^2e^{x^2y}$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = xe^{x^2y}(2ydx + xdy)$$

2. 设 $f(x, y) = x \ln(xy)$, 则 $f'_x(1, e) =$ _____

解 $f'_x(x, y) = \ln(xy) + x \frac{y}{xy} = \ln(xy) + 1$

则 $f'_x(1, e) = 2$

或 $f(x, y) = x \ln(xy) = x \ln x + x \ln y$

$$f'_x(x, y) = \ln x + 1 + \ln y$$

8.2.一

3. 设 $z = \sin(xy)$, 则 $dz =$ _____

$$\text{解 } \frac{\partial z}{\partial x} = y \cos(xy) \quad \frac{\partial z}{\partial y} = x \cos(xy)$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \cos(xy)(ydx + xdy)$$

4. 设 $z = x^3y - xy^3$, 求 $\frac{\partial^2 z}{\partial x \partial y} =$ _____

$$\text{解 } \frac{\partial z}{\partial x} = 3x^2y - y^3$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3x^2 - 3y^2$$

8.2.二

1. 求曲线 $\begin{cases} z = \frac{x^2 + y^2}{2} \\ y = 1 \end{cases}$ 在点(1,1,1)处的切线对于x轴正向所成的倾角 α .

解 $z = \frac{x^2 + y^2}{2} \quad \frac{\partial z}{\partial x} = x$

$$\tan \alpha = \left. \frac{\partial z}{\partial x} \right|_{(1,1,1)} = 1$$

$$\alpha = \frac{\pi}{4}$$

8.2.二

2. 设 $z = 2\cos^2(x - \frac{y}{2})$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解 $\frac{\partial z}{\partial x} = 4\cos(x - \frac{y}{2})[-\sin(x - \frac{y}{2})] = -2\sin(2x - y)$

$$\frac{\partial z}{\partial y} = 4\cos(x - \frac{y}{2})[-\sin(x - \frac{y}{2})](-\frac{1}{2}) = \sin(2x - y)$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \sin(2x - y)(-2dx + dy)$$

8.2.二

3. 设 $z = x \ln(x + y)$, 求其二阶偏导数

解
$$\frac{\partial z}{\partial x} = \ln(x + y) + \frac{x}{x + y} \qquad \frac{\partial z}{\partial y} = \frac{x}{x + y}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x + y} + \frac{x + y - x}{(x + y)^2} = \frac{x + 2y}{(x + y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x + y} + \frac{-x}{(x + y)^2} = \frac{y}{(x + y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-x}{(x + y)^2} \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{x + y - x}{(x + y)^2} = \frac{y}{(x + y)^2}$$

8.2.二

4. 设 $z = x^3 \sin y - ye^x$, 求 $\frac{\partial^2 z}{\partial x \partial y}$.

解 $\frac{\partial z}{\partial x} = 3x^2 \sin y - ye^x$ $\frac{\partial^2 z}{\partial x \partial y} = 3x^2 \cos y - e^x$

5. 设 $z = \arctan \frac{y}{x}$, 求 dz .

解 $\frac{\partial z}{\partial x} = \frac{1}{1 + (\frac{y}{x})^2} (-\frac{y}{x^2}) = -\frac{y}{x^2 + y^2}$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + (\frac{y}{x})^2} (\frac{1}{x}) = \frac{x}{x^2 + y^2}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{x}{x^2 + y^2} (-ydx + xdy)$$

8.2.二

6. 设 $u = a^{x+yz} - \ln x^a (a > 0)$, 求 du .

解 $u = a^{x+yz} - a \ln x$

$$\frac{\partial u}{\partial x} = a^{x+yz} \ln a - \frac{a}{x}$$

$$\frac{\partial u}{\partial y} = za^{x+yz} \ln a$$

$$\frac{\partial u}{\partial z} = ya^{x+yz} \ln a$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

$$= (a^{x+yz} \ln a - \frac{a}{x}) dx + a^{x+yz} \ln a (z dy + y dz)$$

8.2.二

7. 设 $f(t)$ 为连续函数, $u = xyz + \int_{yz}^{xy} f(t)dt$, 求 du

$$\text{解 } \frac{\partial u}{\partial x} = yz + yf(xy) \quad \frac{\partial u}{\partial y} = xz + xf(xy) - zf(yz)$$

$$\frac{\partial u}{\partial z} = xy - yf(yz)$$

$$\begin{aligned} du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \\ &= (yz + yf(xy))dx \\ &\quad + (xz + xf(xy) - zf(yz))dy \\ &\quad + (xy - yf(yz))dz \end{aligned}$$

8.2.二

8. 设 $f(x, y, z) = (\frac{x}{y})^z$, 求 $df(1, 1, 1)$

$$\text{解 } \left. \frac{\partial f}{\partial x} \right|_{(1,1,1)} = z \left(\frac{x}{y} \right)^{z-1} \cdot \frac{1}{y} \Big|_{(1,1,1)} = 1$$

$$\left. \frac{\partial u}{\partial y} \right|_{(1,1,1)} = z \left(\frac{x}{y} \right)^{z-1} \cdot \left(-\frac{x}{y^2} \right) \Big|_{(1,1,1)} = -1$$
$$\left. \frac{\partial u}{\partial z} \right|_{(1,1,1)} = \left(\frac{x}{y} \right)^z \ln \left(\frac{x}{y} \right) \Big|_{(1,1,1)} = 0$$

$$\begin{aligned} df(1, 1, 1) &= \left. \frac{\partial u}{\partial x} \right|_{(1,1,1)} dx + \left. \frac{\partial u}{\partial y} \right|_{(1,1,1)} dy + \left. \frac{\partial u}{\partial z} \right|_{(1,1,1)} dz \\ &= dx - dy \end{aligned}$$

8.2.二

9. 讨论函数 $f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & x^2 + y^2 \neq 0 \\ 0, & x^2 + y^2 = 0 \end{cases}$

在点(0,0)的可微性.

解 $\left. \frac{\partial f}{\partial x} \right|_{(0,0)} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0 - 0}{\Delta x} = 0,$

$$\left. \frac{\partial f}{\partial y} \right|_{(0,0)} = \lim_{\Delta y \rightarrow 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{0 - 0}{\Delta y} = 0$$

$$\Delta f = f(\Delta x, \Delta y) - f(0, 0) = \frac{\Delta x \cdot \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

由于 $\lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta z - [f_x(0, 0) \cdot \Delta x + f_y(0, 0) \cdot \Delta y]}{\rho} = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta x \cdot \Delta y}{(\Delta x)^2 + (\Delta y)^2}$

不存在, 函数 $f(x, y)$ 在(0,0)点不可微.

8.2.三

1. 设 $z = \frac{y}{f(x^2 - y^2)}$, 其中 $f(u)$ 可导, 证明: $\frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} = \frac{z}{y^2}$

证 设 $u = x^2 - y^2$, 则 $z = \frac{y}{f(u)}$

$$\frac{\partial z}{\partial x} = -\frac{y}{f^2(u)} \cdot f'(u) \cdot 2x = -\frac{2xyf'(u)}{f^2(u)}$$

$$\frac{\partial z}{\partial y} = \frac{f(u) - yf'(u) \cdot (-2y)}{f^2(u)} = \frac{1}{f(u)} + \frac{2y^2 f'(u)}{f^2(u)}$$

$$\begin{aligned} \frac{1}{x} \frac{\partial z}{\partial x} + \frac{1}{y} \frac{\partial z}{\partial y} &= -\frac{2yf'(u)}{f^2(u)} + \frac{1}{yf(u)} + \frac{2yf'(u)}{f^2(u)} \\ &= \frac{1}{yf(u)} = \frac{z}{y^2} \end{aligned}$$

8.2.三

2. 设 $r = \sqrt{x^2 + y^2 + z^2}$, 证明: $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}$

$$\text{证 } \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}},$$

$$\frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{\sqrt{x^2 + y^2 + z^2} - \frac{x^2}{\sqrt{x^2 + y^2 + z^2}}}{x^2 + y^2 + z^2}$$

$$= \frac{y^2 + z^2}{\sqrt{(x^2 + y^2 + z^2)^3}}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2 + z^2}{\sqrt{(x^2 + y^2 + z^2)^3}},$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{x^2 + y^2}{\sqrt{(x^2 + y^2 + z^2)^3}}$$

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2(x^2 + y^2 + z^2)}{\sqrt{(x^2 + y^2 + z^2)^3}} = \frac{2}{r}$$